Computational Surrogates for Crumpled Sheets

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The Physics behind Paper Crumpling

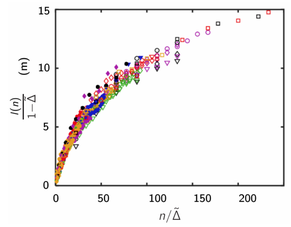
Crease structures in crumpled paper help us understand damage accumulation in mechanical systems. Modern crumple theory is interested in the rules of crumpling and generalizing these to find universality in other complex, disordered systems.

Crumple theory is applicable to the crumpling of graphene films used in batteries [7], the folding of Earth's crust [2], and developing ultrathin electronic devices [5].

In 2018, [3] found empirically that the total crease length l grows logarithmically with the number of crumples n across varying compaction ratios $\tilde{\Delta} \in [0.045, 0.9]$:

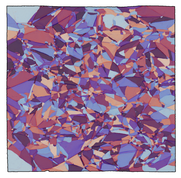
$$l(n) = c_1(1-\tilde{\Delta})\log(1+\frac{c_2 n}{\tilde{\Delta}}) \quad (1)$$

where c_1, c_2 are fitting constants. How can we explain this mathematically simple result?



(From [3]) A logarithmic collapse of all $l(n)$ curves. Different colors and symbols indicate different experimental trials.

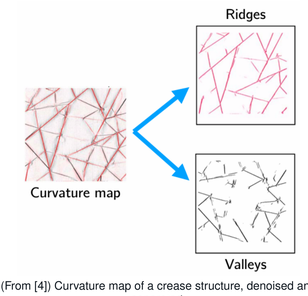
We can shift focus to the **total facets**:



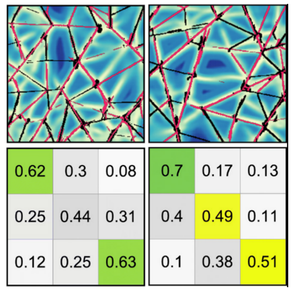
A correspondence between crumpling and fragmentation theory was found and provided a physical explanation for the logarithmic scaling. This was made possible because there was enough data to statistically analyze: 507 high-resolution crease images [1].

Machine Learning in a Data-Limited Regime

Machine learning (ML) models are a popular data analysis tool, but its results are difficult to interpret. *How can we extract scientific insight for crumpling from the results of ML models*?



(From [4]) Curvature map of a crease structure, denoised and separated.



(From [4]) Two ridge predictions using the ML model trained with *only* experimental data, against the actual scans, and their confusion matrices. The upper left entry, for instance, is the probability of correctly predicting regions closest to ridges.

Given valleys, can a convolutional neural network be trained to predict mountains?

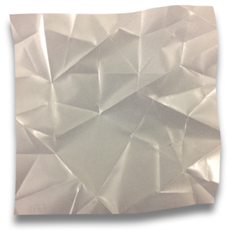
- In 2019, Hoffman *et al.* [4] input the experimental crumple data into a ML model. However, 507 crease map images were insufficient for making accurate predictions.
- How can we uncover the rules of crumpling *in a data-limited regime*?

Flat-Folded Sheets as Computational Surrogates

Initial attempts to make accurate predictions via the ML model were unsuccessful [4]. Simple computational surrogates, **flat-folded sheets**, augmented with the crumple data, were essential for making accurate predictions.

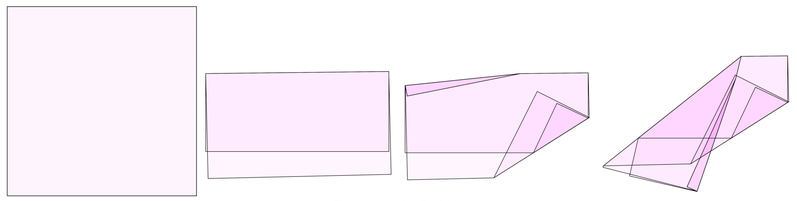
Problem:
Experimental flat-folding data is expensive to gather.
Solution:
Flat-fold *in silico* to produce practically unlimited data!

- Flat-folding is effectively a constrained crumpling process.
- Flat-folding rules are better understood.



A flat-folded Mylar sheet.

The `Voro++` software library efficiently manipulates polygonal facets [6] and the simulations naturally follow flat-folding rules.



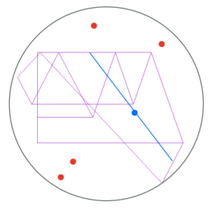
Four successive flat folds.

A Theoretical Question: Random Flat Folding

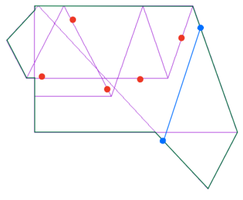
What does it mean to randomly fold a sheet? This question may seem obvious, but one can propose multiple random fold protocols which yield different creasing statistics.

A fold is defined by the angle of its normal vector $\hat{n} = (\cos\theta, \sin\theta)$ and a displacement d in the direction of $\hat{n}$. The angle is always distributed as $\Theta \sim U[0, 2\pi)$.

- **Protocol 1**. Sample uniformly for a point p within the bounding circle until it falls in the sheet. Then $d = p \cdot \hat{n}$.
- **Protocol 2**. Sample for p by weighting facets via area:
  Choose $x \in [0, A)$, where A is the area of the sheet. If A_k is the area of the k^{th} facet and
  $$x \le \sum_{k=1}^{j} A_k \quad (2)$$
  then choose p in the j^{th} facet.
  - Effectively equivalent to picking a point on the original sheet.
- **Protocol 3.** Choose θ first.
  Find the extremal vertices: $a := \min_i \{v_i \cdot \hat{n}\}$, $b := \max_i \{v_i \cdot \hat{n}\}$.
  d is then a realization of $D \sim U(a, b)$.
  - a, b depend on θ, so the distribution D is not uniform across $(-c_r, c_r)$ where c_r is the radius of the bounding circle.
- **Protocol 4.** Choose θ first and, now, $D \sim U(-c_r, c_r)$.
- **Protocol 5**. Choose two points along the edges of any facet until the edge is a boundary of the sheet.

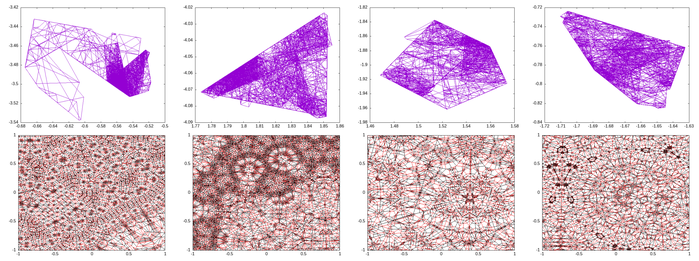


Sampling points within the bounding circle to define the fold.



Sampling points along the edges until they are on the silhouette of the sheet.

Statistical Results of Random Folds

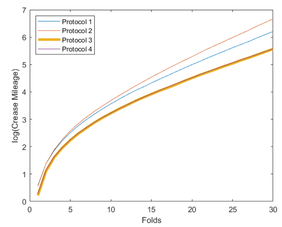


Left to right: fold protocols one through four. Top: the folded sheets after thirty folds. Bottom: the ridge (red) and valley (black) creases of the unfolded sheets in the top row.

Amazingly, the first four fold protocols yield different statistical results, even though they all randomly fold the sheet.

The semi-log graph illustrates the scaling of the total facet counts of the sheets after thirty folds for the first four fold protocols.

- The second protocol yields a growth rate of facets which is significantly higher.
- The third and fourth protocols follow the same scaling.



The scaling of the crease mileage for the first four protocols.

The statistical variance of the facet count can be quite large:

	Fold 1	Fold 2	Fold 3	Fold 4
Max Facets	134,986	357,017	46,635	42,658
Min Facets	2,623	2,277	592	772

Toward New Computational Surrogates

[4] has discovered connections between crumpling and flat-folding, which moves us toward uncovering an order amid the seemingly complex process of crumpling. Next, we would like to study computational surrogates for crumpled sheets of different materials and initial geometries, and sheets crumpled via protocols other than by a piston. We suspect the logarithmic scaling applies broadly.

References

[1] Jovana Andrejevic et al. "A model for the fragmentation kinetics of crumpled thin sheets". In: *Nature Communications* 12.1 (2021), p. 1470. DOI: 10.1038/s41467-021-21625-2.

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[4] Jordan Hoffmann et al. "Machine learning in a data-limited regime: Augmenting experiments with synthetic data uncovers order in crumpled sheets". In: *Science Advances* 5.4 (2019). DOI: 10.1126/sciadv.aau6792.

[5] M. Kaltenbrunner et al. "An ultra-lightweight design for imperceptible plastic electronics". In: *Nature* 499 (2013), pp. 458–463. DOI: 10.1038/nature12314.

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[7] J. Zang et al. "Multifunctionality and control of the crumpling and unfolding of large-area graphene". In: *Nature Matter* 12 (2013), pp. 321–325. DOI: 10.1038/nmat3542.