Random Walks and the Winding Problem: Math 704 Final Project

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Originality Statement

I certify that this is my own work and that all sources used from the internet, literature, or otherwise have been cited.

Ann Bigelow, May 2025

The lesson of Lord Rayleigh's solution is that in open country the most probable place of finding a drunkard man who is at all capable of keeping on his feet is somewhere near his starting point.

Karl Pearson, 1905

1 Introduction

In 1905, Karl Pearson asked in a letter to *Nature*'s readers for the probability that a drunken man, beginning at the origin, arrives between a distance r and $r + dr$ from his starting point after n steps of unit length. Lord Rayleigh answered the question in just one week for the case where n is large. Rayleigh's result was

$$P_n(r) = \frac{2r}{n} e^{-r^2/n} \tag{1}$$

where $P_n(r)dr$ is the probability that the walker travels between r and $r + dr$. This is an early version of the modern Central Limit Theorem in probability. Also in 1905, Einstein published his work on Brownian motion, apparently without knowing of Pearson's problem even though he modeled the continuous Brownian process as a discrete random walk [O1]. A Brownian motion may mathematically be considered as a continuous but nowhere differentiable curve, so considering the Brownian motion as a random walk with small enough steps can be accurate.

Interesting real-world problems are suitably modeled by random walks. Since Einstein and Pearson's work in the early 1900s, the topic of random walks has bloomed into a rich field of study with broad applications and relevance to studies involving Brownian motions and partial differential equations. When its steps are taken from a normal distribution, the random walk can model stock ticks and underlies much of modern theoretical finance, as introduced by Louis Bachelier in 1900. Other interesting applications include the motion of microorganisms (Nossal 1983), random migrations (Pearson, 1906), and superpositions of waves (Slack, 1946). For further discussions of the history of random walks, B. D. Hughes' book, *Random Walks and Random Environments* [1], is quite entertaining.

Figure 1 illustrates an example of a random walk as proposed by Pearson. Here, the green and red dots indicate where the walker began and ended, respectively, after $N = 1000$ steps.

The second, but no less interesting, focus of this project is on the **winding problem**. When we instead shift focus from the probability distribution of the walker's distance from its starting point to the angular component $\Theta(n)$ of its final position after n steps, the "winding" question is introduced. Here, the domain of Θ is $(-\infty, \infty)$ to capture the walker's winding about the

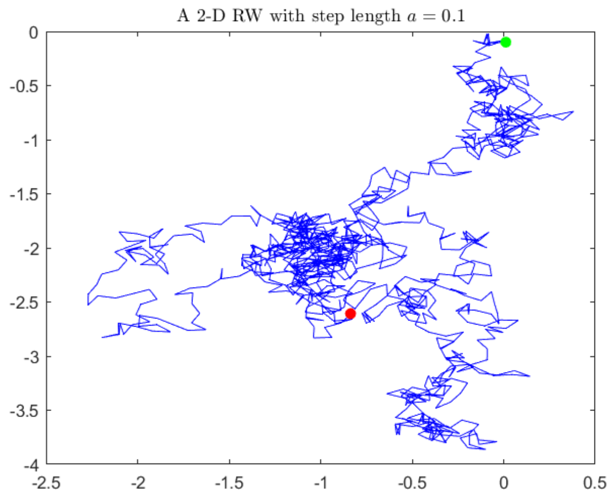

Figure 1: Pearson's Random Walk

point where it began. This is useful, for instance, when studying polymer entanglements (Edwards and Doi, 1988) in the dynamics of polymer solutions.

1.1 Structure of the Project

The aim of this report is to follow some thread which links the theory of random walks, Brownian motions, diffusion, and the winding problem altogether.

For the sake of clarity, I'll now provide some sense of direction. On the broad scale, I will first discuss heuristics relating to random walks and diffusion. This will be followed with a discussion of the winding problem. More specifically:

- First, Einstein's derivation of the diffusion equation will be discussed.
- The Central Limit Theorem will then be derived to arrive at Rayleigh's result given by equation (1). Both Einstein's derivation and this more modern derivation of Rayleigh's result begin by considering the probability of finding a walker at a distance $r + dr$ from its starting point, but both are interesting enough to include. Numerical approximations of Rayleigh's result will then be demonstrated, both for Pearson's walk with unit step length and when the steps are taken from a normal distribution.
- The winding problem will then be introduced. Here, Spitzer's Law will be re-derived in some detail. The lens of the project will necessarily shift to solutions of the diffusion equation where complications are introduced due to the infinite domain of $\Theta(n)$.
- Lastly, numerical approximations of Spitzer's result and its associated MATLAB code will be discussed. All code for the project is publicly available on GitHub.

2 Random Walks and Diffusion

2.1 Einstein's Derivation of the Heat Equation

In his famous paper *Investigations on the Theory of the Brownian Movement* [2] in 1905, Einstein derived the heat equation and discussed the corresponding diffusion coefficient in terms of physical quantities. To do this, Einstein considered a density of particles undergoing Brownian motions for a location x and time t, and their corresponding increments $x + \Delta, t + \tau$.

Norbert Wiener has shown rigorously that a continuous-time Brownian motion in $\mathbb{R}^D$ can be seen as a random walk on a D-dimensional lattice $\mathbb{Z}^D$ in appropriate limits for the lattice and time spacing. Hence, the term "Brownian motion" is appropriately synonymous with a *Wiener process* [3]. The following derivation therefore approximates the Brownian process as a random walk. In this way, some of the initial assumptions made in this derivation resemble the ones included below in the derivation of Rayleigh's result for the Pearson walk.

In one space dimension, Einstein considered $\rho(x,t)$, the density of Brownian particles per unit of space x at time t. Additionally, $\phi(\Delta)$ is defined to be the probability density function for an increment in space Δ.

$\rho(x, t+\tau)$ can be expanded for a small τ using a Taylor series,

$$\rho(x, t+\tau) = \rho(x,t) + \tau \frac{\partial \rho}{\partial t} + O(\tau)^2, \tag{2}$$

but also

$$\rho(x, t+\tau) = \int_{-\infty}^{\infty} \rho(x+\Delta, t)\phi(\Delta) d\Delta \tag{3}$$

by "definition of $\phi(\Delta)$" [2]. It may be helpful to interpret this as a Chapman-Kolmogorov equation.

We can also expand $\rho(x+\Delta, t)$ as

$$\rho(x+\Delta, t) = \rho(x,t) + \Delta \frac{\partial \rho}{\partial x} + \frac{1}{2}\Delta^2 \frac{\partial^2 \rho}{\partial x^2} + \cdots . \tag{4}$$

Thus, from equations (2), (3), and (4),

$$\begin{aligned} \rho(x,t) + \tau\frac{\partial \rho}{\partial t} + O(\tau^2) &= \rho(x,t)\int_{-\infty}^{\infty}\phi(\Delta)d\Delta + \frac{\partial \rho}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta + \frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta + \cdots \\ &= \rho(x,t)\cdot 1 - 0 + \frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta + \cdots \end{aligned}$$

The particle is equally likely to move right as left, so the expectation of Δ, $\int_{-\infty}^{\infty} \Delta\phi(\Delta)d\Delta$, vanishes because the integrand is odd. This reasoning may be applied for all of the terms with odd derivatives in x; the odd moments vanish due to spatial symmetry.

Therefore,

$$\frac{\partial \rho}{\partial t} \sim \frac{\partial^2 \rho}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2\tau} \phi(\Delta)d\Delta, \tag{5}$$

after ignoring higher-order terms in Δ. This simplification is physically sound, as the "steps" in a Brownian motion are small.

If we allow

$$\gamma := \int_{-\infty}^{\infty} \frac{\Delta^2 \phi(\Delta)}{2\tau} d\Delta,$$

equation (5) is the diffusion equation we very well know and love,

$$\frac{\partial \rho}{\partial t} = \gamma \nabla^2 \rho \tag{6}$$

for the density of Brownian particles $\rho(x,t)$. Notably, the diffusion coefficient is proportional to the mean-squared displacement $\langle \Delta^2 \rangle$ of the Brownian particle. As such, in this case we require it to be finite:

$$\int_{-\infty}^{\infty} \Delta^2 \phi(\Delta) d\Delta < \infty.$$

2.1.1 Solving the Diffusion Equation

For the random walk, the equation's initial condition is

$$\rho(x,0) = \delta(x) \tag{7}$$

to ensure that the walk begins at the origin. The diffusion equation may then be solved directly using the Fourier and inverse Fourier transforms

$$\hat{\rho}(k,t) = \int e^{-ikx} \rho(x,t) dx, \;\; \rho(x,t) = \frac{1}{2\pi} \int e^{ikx} \hat{\rho}(k,t) dk. \tag{8}$$

(6) then becomes

$$\frac{\partial \hat{\rho}}{\partial t} = -\gamma k^2 \hat{\rho}(k,t)$$

which is effectively an ODE with the very nice solution

$$\begin{aligned}\hat{\rho}(k,t) &= e^{-\gamma k^2 t}\hat{\rho}(k,0) \\ &= e^{-\gamma k^2 t}\int e^{-ikx}\delta(x)dx \\ &= e^{-\gamma k^2 t}\end{aligned}$$

so that

$$\rho(x,t) = \frac{1}{2\pi}\int e^{ikx-\gamma k^2 t}dk \tag{9}$$

$$\implies \rho(x,t) = \frac{n}{\sqrt{4\pi\gamma t}}e^{-x^2/4\gamma t} \tag{10}$$

where n is the number of Brownian particles in the system of interest. This is exactly the expression that describes a normal distribution of the random variable X representing the position of a Brownian particle with mean 0.

From here, Einstein concludes the variance is the mean squared displacement, given by

$$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] = 2\gamma t.$$

Importantly, this means that the displacement of the Brownian particle is proportional to the square root of the time elapsed [3]. In this case, where γ exists, we have *normal diffusion*; otherwise, the diffusion may be *anomalous* [O1]. A discussion of anomalous diffusion is beyond the scope of this review, but the MIT course on random walks and diffusion provides is a good place to start [O1].

In the next section, we'll see that the distribution of the discrete walker's final distance (equation (17)) is the same, with $n = 1$ and $t = N\tau$.

2.2 Deriving Rayleigh's Solution: The Central Limit Theorem

The Central Limit Theorem states that the asymptotic probability density of the position of a random walker after N steps is

$$P_N(x) \to \frac{1}{\sqrt{2\pi N\sigma^2}}e^{-(x-N\mu)^2/2N\sigma^2} \tag{11}$$

where $N \to \infty$. The steps themselves are independent and identically distributed with finite mean μ and variance σ^2, and are drawn from the distribution $p(x) = p(x|0)$. We will derive this by following the notes from [O2] and then arrive at Rayleigh's result by following [O1].

The following derivation can be generalized for higher dimensions, but for the sake of simplicity we will work in one dimension.

The probability distribution for the position of the walker after N independent and identically-distributed steps is described by $P_N(x)$, with $P_0(x) = \delta(x)$. The Chapman-Kolmogorov equation gives us the recursion

$$P_{N+1}(x) = \int P_N(x')p(x|x')dx'. \tag{12}$$

This simply states that the walker must first arrive at a location x' in N steps and then transition from x' to x with probability $p(x|x')$ to get to x in $N+1$ steps.

Because the right-hand side of (12) is the convolution $\{P * p\}(x)$, take its Fourier transform, defined in this case to be

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx; \;\; f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}dk, \tag{13}$$

and use the Fourier convolution theorem to arrive at

$$\hat{P}_{N+1}(k) = \hat{P}_N(k)p(k) \tag{14}$$

$$\implies \hat{P}_N(k) = \hat{P}_0(k)(p(k))^N. \tag{15}$$

The transformed initial condition, which ensures the walker begins at the origin, is

$$\begin{aligned}\hat{P}_0(k) &= \int \delta(x)e^{ikx}dx \\ &= e^0 = 1.\end{aligned}$$

Equation (15) then becomes

$$\hat{P}_N(k) = (p(k))^N. \tag{16}$$

To take the inverse Fourier transform of (16), first put

$$\begin{aligned}p(k) &= \int_{-\infty}^{\infty} p(x)e^{ikx}dx \\ &= \int_{-\infty}^{\infty} p(x)[1 + ikx - \tfrac{1}{2}k^2x^2 + \cdots]\,dx \\ &= 1 + ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle + \cdots\end{aligned}$$

using the series formulation of e^x. Equation (16) becomes

$$P_N(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{N\ln(1+ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle)} e^{-ikx} dk.$$

The logarithmic term in the exponent can be approximated using the Mercator series $\ln(1+\epsilon) \approx \epsilon - \frac{\epsilon^2}{2} + \cdots$ when the first and second moments of $p(x)$ are finite and small,

$$\begin{aligned}\ln(1+ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle) &\approx ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle - \frac{1}{2}\Big(ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle\Big)^2 \\ &\approx ik\langle x\rangle - \frac{1}{2}k^2\Big(\langle x^2\rangle - \langle x\rangle^2\Big) \\ &= ik\mu - \frac{1}{2}k^2\sigma^2\end{aligned}$$

where $\langle x\rangle = \mu$ and the variance of the steps is σ^2.
Hence,

$$P_N(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{N(ik\mu - \frac{k^2}{2}\sigma^2)} e^{-ikx} dk.$$

This integral can now be computed explicitly by completing the square in k in the exponent and using $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$. Finally, we arrive at the Central Limit Theorem in the desired form,

$$P_N(x) \sim \frac{1}{\sqrt{2\pi N\sigma^2}} e^{-(x-N\mu)^2/2N\sigma^2} \tag{17}$$

for finite μ and σ^2.
Rayleigh's probability density is extracted from (17) when we change to polar coordinates,

$$P_N(r) = A_D r^{D-1} P_N(\mathbf{x}), \tag{18}$$

where A_D is the surface area of the D-dimensional unit sphere A_D and r is the distance from the origin. Now that we have lifted to D dimensions, it is relevant to mention that Pearson's random walk is *isotropic*, meaning $p(\mathbf{x}) = p(r)$, $r = |\mathbf{x}|$, and $\mu = 0$ since any stochastic drift is negligible. Thus,

$$P_N(r) \sim \frac{2r}{\sigma^2 N} e^{-r^2/\sigma^2 N} \tag{19}$$

for $D = 2$. This is the formula that is used for comparison in the code `rw.m`. For a fixed step length a, as in Pearson's random walk, $p(r) = \delta(r-a)$ and

$$\sigma^2 = \int_{-\infty}^{\infty} r^2 p(r) dr = a^2.$$

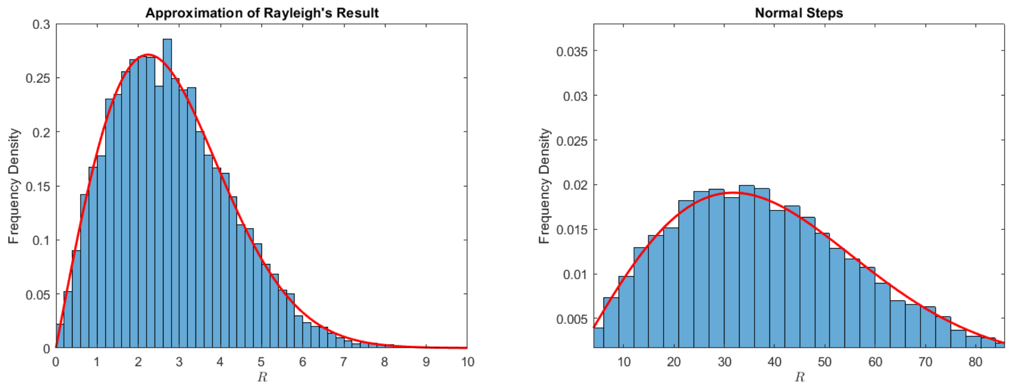

Figure 2: Comparison to Rayleigh's Result with a fixed step length (left) and normally-distributed steps (right). The approximation is a histogram in blue and the exact result is in red.

$P_N(r) \sim \frac{2r}{a^2N}e^{-r^2/a^2N}$ is exactly Lord Rayleigh's solution for $a = 1$.

The MATLAB code `rw.m` performs a random walk and compares approximations of Rayleigh's distribution against the exact result (see figure 2).

3 Introduction to the Winding Problem

The winding problem, introduced by Lévy (1940), investigates a two-dimensional random walk expressed in polar coordinates. The variables R_n, Θ_n describe the radius and angle of the walker after n steps, where $\Theta_n \in \mathbb{R}$ and $R_n \geq 0$. The total, unwrapped angular component captures *winding* around the walker's starting point.
Conventionally, $\Theta_1 = 0$ so that the walker's first step defines the angular reference. This allows for symmetry of the angular displacement, $\mathbb{E}[\Theta_n] = 0$, as the walker is equally likely to move clockwise as counterclockwise [1].

In 1989, Claude Belisle found that the winding angle for a random walk with independent, identically distributed steps exhibits a hyperbolic secant distribution as $n \to \infty$:

$$\frac{2\Theta_n}{\log n} \to X;\ p_X(x) = \frac{1}{2}\text{sech}\left(\frac{\pi x}{2}\right) \tag{20}$$

whereas, for a Brownian motion in the plane, Frank Spitzer showed in 1958 that the winding angle $\Theta(t)$ is asymptotically distributed as a Cauchy law as $t \to \infty$:

$$\frac{2\Theta(t)}{\log t} \to X;\ p_X(x) = \frac{1}{\pi(1 + x^2)} \tag{21}$$

by considering the problem in a wedge and taking an infinite limit of the wedge angle [4]. This is *Spitzer's Law*.

For the continuous Brownian process, the positive integer moments of $\Theta(t)$ all diverge. However, the same moments of the discrete random walk are all finite – in this case, the walker may not get infinitely close to the origin by nature of its walk.

According to Comtet *et al.* [5], Messulam and Yor (1982) split the winding angle $\Theta(t)$ of the Brownian motion into its big and small components, Θ_+ and Θ_-, to capture winding outside and inside the unit disk separately. The large winding angle was found to obey

$$\frac{2\Theta_+}{\log t} \to X_+; \; p_{X_+}(x_+) = \frac{1}{2}\text{sech}\left(\frac{\pi x_+}{2}\right). \tag{22}$$

This is Belisle's result in equation (20). Thus, the small winding angle is responsible for the discrepancy between (20) and (21); the divergence of the moments of $\Theta(t)$ is attributed to "the roughness of the Brownian trajectory generating large winding near the origin," as put nicely by Wen and Thiffeault [4]. This behavior is disadvantageous when modeling physical phenomena. Working with a Brownian-diffusion process outside of the unit disk thus avoids the issue. Other motions, like a *self-avoiding* one which never crosses its path, also yield finite moments [5].

3.1 Derivation of Spitzer's Law

In this section, the diffusion equation

$$\frac{\partial P}{\partial t} = \nabla^2 P(\mathbf{x}, t) \tag{23}$$

will be solved for $P(\mathbf{x}, t)$, the probability density for finding the Brownian particle in a small region around the spatiotemporal point $(\mathbf{x}, t)$ with $\mathbf{x} \in \mathbb{R}^2$. Unsurprisingly, the initial condition for the problem is

$$P(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x_0})$$

to be certain that the Brownian particle begins at $\mathbf{x_0}$.

The diffusion coefficient here is $\gamma = 1$, without loss of generality – time re-scalings can be done for any problem to achieve this, as we have discussed in class.

To find the distribution of the winding angle, the spatial domain of the problem is taken to be a disk's wedge of angle 2α, centered horizontally. The problem will be converted into polar coordinates (r, θ), $\theta \in (-\alpha, \alpha)$, and the limit $\alpha \to \infty$ will be taken.

In polar coordinates, equation (23) is

$$\partial_t P = \partial_{rr} P + \frac{1}{r}\partial_r P + \frac{1}{r^2}\partial_\theta P, \quad P(r,\theta,0) = \frac{1}{r}\delta(r-r_0)\delta(\theta) \tag{24}$$

with zero-Dirichlet boundary conditions at the walls of the wedge, as specified in [4]. For now, this problem concerns winding around a point, so $r \in (0, \infty)$. Other cases, like winding in an annulus or around a disk, are discussed in [4].

The separation of variables method will be used to begin. Because this is a project for a course, more details are included here than in the literature. The work was checked against publicly-available notes from the Bioengineering Department at UCSD, [O3].

Assume that P is separable, so it has the form

$$P(r,\theta,t) = R(r)\Theta(\theta)T(t). \tag{25}$$

Unfortunately, this Θ does *not* describe the distribution of the winding angle. Substituting (25) into (24) yields

$$\frac{T'}{T} = \frac{1}{r}\frac{R'}{R} + \frac{R''}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}. \tag{26}$$

Primes are appropriate to denote derivatives here because each function involved is dependent on a single variable. (26) states that a function of t is identically equal to one of r, θ. As such, the entire equation may be set equal to a conveniently-named constant, $-\lambda^2$, as is standard for the separation of variables technique.
From the left function,

$$T'(t) = -\lambda^2 T \implies T(t) = e^{-\lambda^2 t}$$

and the right function yields

$$r\frac{R'}{R} + r^2\frac{R''}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta}. \tag{27}$$

Again, set (27) equal to a constant, μ^2, to produce two equations,

$$\begin{cases} \Theta'' + \mu^2\Theta = 0 \\ r^2 R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0. \end{cases}$$

The equation for $\Theta(\theta)$ has sine and cosine solutions, or one of the form $e^{i\mu\theta}$.
On the other hand, by introducing the change of variables

$$r = \frac{x}{\lambda}; \;\; R'(r) = \frac{d\hat{R}}{dx}\frac{dx}{dr} = \lambda \hat{R}'(x),$$

the radial equation may be transformed into

$$x\hat{R}' + x^2\hat{R}'' + (x^2 - \mu^2)\hat{R} = 0,$$

which is Bessel's equation. The solution $\hat{R}$ is a Bessel function of the first kind, the "J" kind,

$$\hat{R}(x) = J_\mu(x) \implies R(r) = J_\mu(\lambda r).$$

Thus, from (25), $P(r, \theta, t)$ takes the form

$$e^{-\lambda^2 t} e^{i\mu\theta} J_\mu(\lambda r). \tag{28}$$

At this point, typically the discrete eigenvalues μ_k and λ_k are summed over, where $k \in \mathbb{Z}$. In these instances the solutions are periodic in θ. This is clearly not the case here. Instead, we will take the limit as $\alpha \to \infty$, so the eigenvalues may exist over a continuum and the summation must be extended to an integral representation.
This process is discussed in detail in J. L. Thiffeault's lecture notes [O4], but a few words about it might be appreciated. Because the sum is taken over the eigenvalues μ_k, the distance between successive μ_k is found in order to specify $d\mu$. $d\mu_k$ is found to be inversely proportional to the wedge angle α. Thus, the increment $d\mu_k$ is ensured to be small in the limit $\alpha \to \infty$. From here, an integral representation over $\mu \in \mathbb{R}_{>0}$ for $P(r, \theta, t)$ is uncovered.

Allow me to jump to the integral representation as written by Wen and Thiffeault [4],

$$P(r, \theta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\lambda^2 t} e^{i\mu\theta} J_{|\mu|}(\lambda r) J_{|\mu|}(\lambda r_0) \lambda \; d\lambda d\mu, \tag{29}$$

for consistency with the separated form in (28). The term $\lambda J_{|\mu|}(\lambda r_0)$ allows for the initial condition to be achieved. It is not immediately obvious why this is, so the following verification may be useful.

A convenient property of the J Bessel functions is

$$\delta(r - r_0) = r \int_0^{\infty} J_{|\mu|}(\lambda r) J_{|\mu|}(\lambda r_0) \lambda d\lambda$$

[6]. This allows us to write

$$P(r,\theta,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu\theta} J_{|\mu|}(\lambda r) J_{|\mu|}(\lambda r_0)\lambda \, d\lambda d\mu$$
$$= \frac{1}{2\pi r}\delta(r-r_0)\int_{-\infty}^{\infty} e^{i\mu\theta} d\mu.$$

The Fourier transform of $\delta(\theta)$ is

$$\mathcal{F}\{\delta(\theta)\} = \int_{-\infty}^{\infty} e^{-i\mu\theta}\delta(\theta)d\theta = 1 =: \hat{\delta}(\mu)$$

so the inverse Fourier transform becomes

$$\delta(\theta) = \mathcal{F}^{-1}\{\mathcal{F}\{\delta(\theta)\}\}$$
$$= \mathcal{F}^{-1}\{\hat{\delta}(\mu)\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu\theta} d\mu.$$

This means $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu\theta} d\mu = \delta(\theta)$, and $P(r,\theta,0) = \frac{1}{r}\delta(r-r_0)\delta(\theta)$, as desired.

From (29), we would like to reveal the distribution of the winding angle $\Theta(t)$ in the limit as $t \to \infty$. The following argument summarizes some of the work in [4] with the modification of a no-drift process.

As t becomes large, Watson's lemma in asymptotics tells us the integral in (29) is dominated by small λ. Then the Bessel function $J_{|\mu|}(\lambda r_0)$ may be approximated as

$$J_{|\mu|}(\lambda r_0) \sim \frac{(\frac{1}{2}\lambda r_0)^{|\mu|}}{\Gamma(|\mu|+1)}$$

using the generalized hypergeometric function ${}_0F_1(1+|\mu|; -\frac{1}{4}(\lambda r_0)^2)$ because $(\lambda r_0)^2 << |1+|\mu||$, so ${}_0F_1 \sim 1 + O((\lambda r_0)^2)$ [7]. The same approximation for the $J_{|\mu|}(\lambda r)$ term cannot be made because r may be large.

(29) then becomes

$$P(r,\theta,t) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\frac{r_0}{2})^{|\mu|}}{\Gamma(1+|\mu|)} e^{i\mu\theta}\int_0^{\infty} e^{-\lambda^2 t}\lambda^{|\mu|+1} J_{|\mu|}(\lambda r)\, d\lambda d\mu.$$

The distribution of the winding angle will be found by considering the marginal distribution for the angle

$$W(\theta,t) = \int_0^\infty P(r,\theta,t)\ rdr.$$

The r, λ integral is done directly in [4]. The details are beyond the scope of this project. From [4],

$$W(\theta,t) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{r_0 e^{\gamma/2}}{2\sqrt{t}}\right)^{|\mu|} e^{i\mu\theta} d\mu$$

where γ is the Euler-Mascheroni constant.
To evaluate this integral, define

$$\begin{aligned}
A(t) &:= \frac{r_0 e^{\gamma/2}}{2}\frac{1}{\sqrt{t}} \sim \frac{1}{\sqrt{t}},\ t\to\infty; \\
B(t) &:= -\log(A(t)) = \log\left(\frac{2\sqrt{t}}{r_0 e^{\gamma/2}}\right) \sim \frac{1}{2}\log t,\ t\to\infty.
\end{aligned}$$

Notice that $B(t) > 0$ for $t \to \infty$, so

$$\begin{aligned}
W(\theta,t) &\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\mu|B(t)} e^{i\mu\theta} d\mu \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{\mu(B+i\theta)} d\mu\ + \int_0^\infty e^{\mu(-B+i\theta)} d\mu\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{B(t)+i\theta} + \frac{1}{B(t)-i\theta}\right) \\
&= \frac{1}{\pi}\frac{B}{B^2+\theta^2}.
\end{aligned}$$

This is nearly Spitzer's Law. Using the fact that $B(t) \sim \frac{\log t}{2}$,

$$W(\theta,t) \sim \frac{2}{\log t}\frac{1}{\pi}\frac{1}{1+(\frac{2\theta}{\log t})^2}.$$

If we define $x = 2\theta/\log t$, we arrive at

$$\frac{2\Theta(t)}{\log t} \to X,\ p_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

so that the winding angle distribution behaves as a scaled standard Cauchy distribution, as desired.

3.2 Numerics

The MATLAB code `spitz_vectorized.m`, available on GitHub, compares the winding angle distribution for a discretized Brownian motion against Spitzer's law. Details about the code below may be of interest.

The code uses a diffusion coefficient $\gamma = 0.0001$ and various values for the number of steps N and step size h.
The walk is simulated by discretizing the stochastic differential equation

$$dX = \sqrt{2D}dW$$

where X, W are two-dimensional vectors and W describes the Brownian "steps," with $\mathbb{E}[W^2] = 1$ and $\mathbb{E}[W] = 0$. This is discretized by transforming the differential equation into

$$X_i = X_{i-1} + \sqrt{2Dh}\, W_i.$$

dW became $\Delta W_i = \sqrt{h}W_i$, where $h = \mathbb{E}[(\Delta W)^2]$. For better approximations, h is small, as we saw above in the discussion of the Central Limit Theorem.
The code makes use of an array `steps`, which is generated using MATLAB's built-in `randn` function, storing $\texttt{N} \times 2 \times \texttt{batch}$ (x, y) coordinates of the steps pulled from a normal distribution.
To plot the distribution for large values of `N`, this array would become quite large and require a considerable amount of memory, were it sized $\texttt{N} \times 2 \times \texttt{trials}$. In fact, if `N` and `trials` are large enough, MATLAB kills the program.
Hence, the code uses batches of trials, where $\texttt{trials} = \texttt{batch} \cdot \texttt{batch\_repeat}$. Here, `batch` walks are completed and the total angle of the final position is collected for each. This process is repeated `batch_repeat` times. Then only one outer loop over the batches is utilized and the code is more efficient than an entirely loop-based algorithm, while still allowing for a large number of trials.

The results were visually worse when the asymptotic version of the logarithmic term in Spitzer's Law, $B(t) \sim \frac{\log t}{2}$, was used. So, instead, the code uses

$$\frac{\Theta(t)}{\log(2\sqrt{t}/(r_0 e^{.577/2}))} \to X,\ p_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \tag{30}$$

as the exact distribution. The constant $\gamma \approx 0.577$ is approximately the Euler-Mascheroni constant.

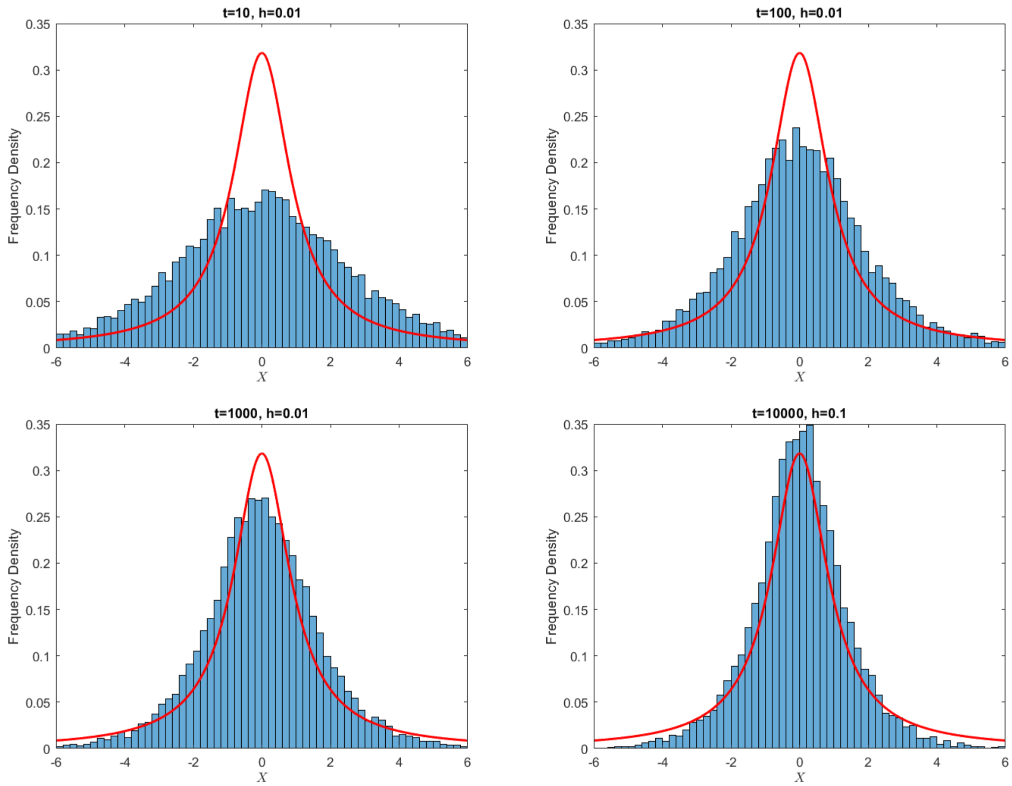

Figure 3: Comparisons of the numerical approximation of Spitzer's probability density against the exact result for varying values of t, h.

3.2.1 Comparing the Numerical Approximations

Figure 3 roughly shows that, as t becomes larger and h smaller, the approximation increases in accuracy in the eyeball norm. I anticipate that the tails of the approximation would lift and its peak would lower if N (and t) were larger and h smaller.

Out of curiosity, figure 4 plots the comparison for a larger t where `r0` is chosen to be 1.5. This looks to be a better fit. I am unsure why a larger choice of `r0` should produce a better result.

Additionally, the code `belisle.mlx` compares the distribution of the winding angle for a random walk against Belisle's hyperbolic secant distribution. 5 illustrates the results.

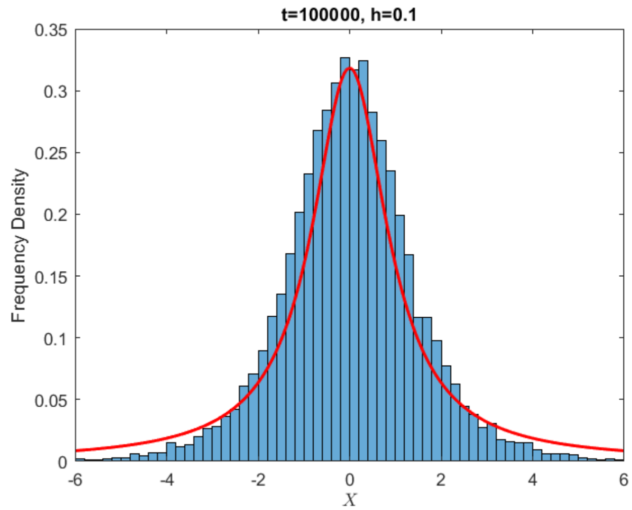

Figure 4: Spitzer's Law against Numerical Approximation, $r_0 = 1.5$

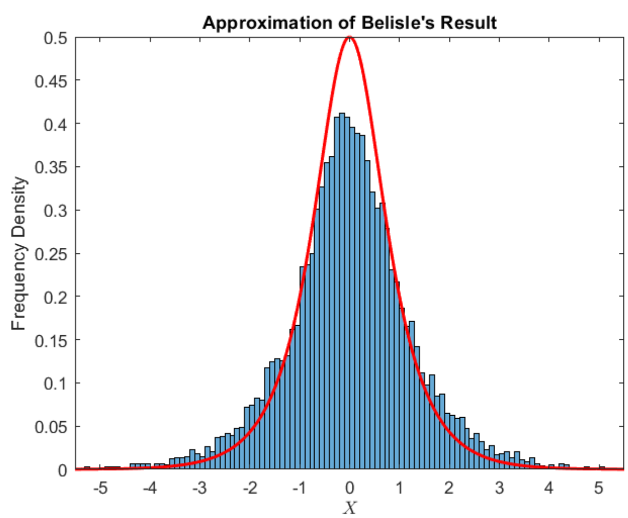

Figure 5: Approximation of Belisle's Law

4 Final Remarks

I have learned a whole lot from this project. Prior to this, I had little to no interest in studying probability or stochastic differential equations, but I now see their importance and underlying beauty.

Math 704 has been a very instructive class and I anticipate that I will use its PDE solution methods, and other applied and computational mathematical tidbits, throughout my career.
Thank you, Professor Thiffeault!

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